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CRITERIA OF THE FIXED SIGN PROPERTY OF HIGHER ORDER FORMS

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We formulate necessary and sufficient conditions for higher order forms to be of fixed sign; these conditions are obtained on the basis of a corresponding theorem of Weierstrass and the method of undetermined Lagrangian multipliers. We determine an algorithm for the covering of a manifold, which makes explicit the fixed sign property of a form. The algorithm presupposes the use of an electronic digital computer.

In studying the stability of motion there is a need for the detection of the fixed sign property of higher order forms [1, 2]. At bifurcation points the Hessian of the potential energy vanishes and it becomes impossible, in principle, to establish an isolated minimum of the potential energy from the lower order terms. A similar case is also mentioned in [3]. It is a known fact (see [4, 5]) that effective general criteria for higher order forms to be of fixed sign do not exist.

1. Let E_m be a real m-dimensional space and let $V(x)$ be a homogeneous form of degree $2n$ (the coefficients $a_{i_1 \ldots i_m}$ are real)

$$V(x) = \sum_{i_1 + \ldots + i_m = 2n} a_{i_1 \ldots i_m} (x^1)^{i_1} \ldots (x^m)^{i_m}$$

Theorem 1. A necessary and sufficient condition for the form $V(x)$ to be positive definite is that it be positive on all the real solutions of the nonlinear system of equations

$$\frac{\partial V(x)}{\partial x^j} - 2\frac{n}{a^2} V(x) x^j = 0, \quad x'x = a^2 \quad (j = 1, \ldots, m) \tag{1.1}$$

($a \neq 0$ is a real number).

Proof. The necessity of the condition is obvious. We consider its sufficiency. Let $V(x)$ be positive on all the real solutions of the system of equations (1.1). Then this form is positive at the element x_* which minimizes the form on the manifold $K = \{x : x'x = a^2\}$. In fact, the real element x_* exists on the basis of Weierstrass' theorem by virtue of the continuity of the function $V(x)$ and the fact that the manifold K is bounded and closed. On the other hand, x_*, along with a specially chosen scalar λ satisfies the system of equations

$$\frac{\partial V(x)}{\partial x^j} + 2\lambda x^j = 0, \quad x'x = a^2 \quad (j = 1, \ldots, m) \tag{1.2}$$

to which the method of undetermined Lagrange multipliers, used to obtain necessary conditions for a conditional extremum, reduces. Multiplying the j-th equation of the system (1.2) by x^j and adding these equations, we obtain, upon taking into account Euler's formula for homogeneous forms, $\lambda = -nV(x) / a^2$. Therefore the element x_* is necessarily a solution of the system of equations (1.1). Thus from the positiveness of the form

$V(x)$ on all the real solutions of the system of equations (1.1) it follows that

$$V(x_*) = \min_{x \in K} V(x) > 0$$

For each $x \in E_m$ we can always select a scalar $\xi \neq 0$ such that $\xi x \in K$. But $V(\xi x) \geqslant V(x_*)$. Therefore $V(x) = V(\xi x) / \xi^{2n} > 0$, Q.E.D.

If $V(x) = x'Ax$ (A is a constant symmetric matrix) and $a = 1$, then the system of equations (1.1) assumes the form

$$Ax - (x'Ax)x = 0, \quad x'x = 1$$

This system has no other solutions besides the normalized eigenvectors h_j of the matrix A, corresponding to the eigenvalues $\varkappa_j$ of this matrix, and $V(h_j) = h_j' A h_j = \varkappa_j$. An application of Theorem 1 for a quadratic form leads to the following well-known fact: a necessary and sufficient condition for a quadratic form to be positive definite is that all the eigenvalues of the matrix A be positive.

The following method is useful, in the general case, for obtaining the set D of all the real solutions of the system of equations (1.1). We consider the subsets D_0 and D_1 of all the real solutions of this system for which $x^1 = 0$ and $x^1 \neq 0$, respectively; $D = D_0 \cup D_1$. Setting $x^1 = 0$ in the equations (1.1), we obtain simpler equations for finding D_0. Finding D_1 reduces to solving the equations

$$x^j \frac{\partial V(x)}{\partial x^1} = x^1 \frac{\partial V(x)}{\partial x^j} \qquad (j = 2, \ldots, m) \tag{1.3}$$

$$\frac{\partial V(x)}{\partial x^1} = 2nx^1 \frac{V(x)}{a^2}, \qquad x'x = a^2$$

Following the change of variables $x^j = k^j x^1$ $(j = 2, \ldots, m)$, the first ($m - 1$) equations are found to be independent of x^1 and can be considered separately. The usefulness of this approach can be demonstrated by applying it to a form of the fourth degree involving two variables. It is easily seen that in fact the problem of detecting the fixed sign property reduces to that of obtaining the roots of an algebraic equation of the fourth degree. We then note the possibility of splitting the set D into subsets D_0 and D_ε for which $x^s = 0$ and $x^s \neq 0$ (s is an arbitrary one of the numbers $1, \ldots, m$).

A second approach for determining D relies on numerical methods. Its theoretical basis is the following simple concept. If the form $V(x)$ is positive at some point x_1, it is positive in some neighborhood of x_1. If the elements of the set D are calculated with sufficient accuracy, then positiveness of the form $V(x)$ on these approximate solutions of the system (1.1) will imply the positiveness of this form on the solutions of the system of equations (1.1). A corresponding estimate of the heighborhood of the point x_1 is given below.

The criterion for the fixed sign property formulated in Theorem 1, can be used in studying more complicated functions in comparison with homogeneous forms, functions for which positiveness on the manifold K assures their positive definiteness. The simplest of such subclass of functions is formed from sums of homogeneous forms of positive terms. Indeed, if

$$V(x) = \sum_{s=0}^{p} V_{2(n+s)}(x)$$

(p is a positive integer and $V_{2(n+s)}(x)$ is a homogeneous form of degree $2(n+s)$), then for each $x \in E_m$ we can find a scalar $\xi \neq 0$ and an integer q such that

$$\xi x \in K, \quad V(x) \geqslant V_{2(n+q)}(x) = V_{2(n+q)}(\xi x) / \xi^{2(n+q)} > 0$$

Here the system of equations analogous to the system (1.1) is somewhat more complicated.

2. In the space E_m we introduce the norm $|x| = \max_{1 \leqslant s \leqslant m} |x^s|$ and we consider the manifold $K = \{x : |x| = a^2\}$.

We consider the following algorithm for detecting the positiveness of a homogeneous form on the manifold K. Let $x_1 \in K$ and assume that $V(x_1) > 0$. We write the relation

$$V(x_1 + \Delta x) - V(x_1) = \sum_{i=1}^{2n} d^i \frac{V(x_1)}{i!} \tag{2.1}$$

where

$$d^i V(x_1) = \left[\left(\frac{\partial}{\partial x^1} dx^1 + \ldots + \frac{\partial}{\partial x^m} dx^m\right)^i V(x)\right]\Bigg|_{x_1} \tag{2.2}$$

On the right side of Eq. (2.2) we have a differential operator obtained by formally raising the expression in parentheses to a power. The differentials appearing therein to the various powers are exactly equal to the increments Δx^j. The differential $d^i V(x_1)$ is a form of degree i in $dx^1, \ldots, dx^m$, whose coefficients are the i-th partial derivatives of $V(x)$ multiplied by "polynomial" constants.

We have the estimate

$$|d^i V(x_1) / i!| \leqslant b_i |\Delta x|^i \quad (i = 1, \ldots, 2n)$$

where for the b_i we can take the largest modulus of the coefficients of the form $d^i V(x_1)$. divided by $i!$ It is clear that

$$|V(x_1 + \Delta x) - V(x_1)| \leqslant \sum_{i=1}^{2n} b_i |\Delta x|^i$$

Let ξ_1 be the root of the equation

$$F(\xi, V(x_1)) = \sum_{i=1}^{2n} b_i \xi^i - V(x_1) = 0 \tag{2.3}$$

Then if $x \in H_1 = \{x : |x - x_1| < \xi_1\}$, it follows that $V(x) > 0$.

We introduce the notation $S_1 = K \cap H_1$, and we denote the closure of S_1 by $\bar{S}_1$. We choose an element from E_m such that $x_2 \in K$ and $x_2 \in \bar{S}_1 \setminus S_1$. Suppose that $V(x_2) > 0$. We find a root ξ_2 of the equation $F(\xi, V(x_2)) = 0$ and a neighborhood $H_2 = \{x : |x - x_2| < \xi_2\}$. Consider the set $S_2 = K \cap (H_1 \cup H_2)$ and its closure $\bar{S}_2$. For $x_3 \in K$ and $x_3 \in \bar{S}_2 \setminus S_2$ we construct the sets H_3, $S_3 = K \cap (H_1 \cup H_2 \cup H_3)$, etc.

D e f i n i t i o n. We say that the manifold K is coverable if a number N exists such that $\bar{S}_N \cap K = K$.

T h e o r e m 2. A necessary and sufficient condition for the positive definiteness of the form $V(x)$ is that the manifold K be coverable.

P r o o f. Necessity of the condition. Positive definiteness of $V(x)$ implies the existence of a number ε such that $\min_{x \in K} V(x) \geqslant \varepsilon > 0$. On the half-open interval $[0, \infty)$ the equation $F(\xi, \varepsilon) = 0$ has a single root. In fact, let us assume the contrary: $\xi' \neq \xi$, where both ξ' and ξ are roots of the equation. Then

$$F(\xi', \varepsilon) - F(\xi, \varepsilon) = \sum_{i=1}^{2n} b_i [(\xi')^i - \xi^i]$$

but this is not possible since the left side of this equation is zero and the expression on the right is either strictly positive or strictly negative. Further it is clear that the root of the equation $F(\xi, \varepsilon) = 0$ increases monotonically as ε increases. But the function $V(x)$ is bounded on K from below; therefore the root $\xi(x)$ of the equation $F(\xi, V(x)) = 0$ is also bounded from below by a certain number $\delta > 0$ depending on ε, i.e. $\xi_{x \in K} \times (x) \geqslant \delta > 0$. Consequently, for realization of the covering algorithm for all i it is found that $\xi_i \geqslant \delta$ $(i = 1, 2, \ldots)$. Assume now that no finite number N exists such that $\bar{S}_N \cap K = K$ and that the process is infinite. Then on the manifold K there exists an infinite divergent sequence $\{x_i \in K\}$, since each element of it is at a distance of at least δ from all the other elements. But the manifold K is a compact subset of a complete metric space [6]. Therefore from the sequence $\{x_i\}$ we can select a convergent subsequence. The contradiction confirms the fact that when the algorithm is realized the manifold K is covered by a finite number of balls and a number N can then be found such that $\bar{S}_N \cap K = K$.

Sufficiency of the condition. If the manifold K is coverable, then $V_{x \in K}(x) > 0$. But since the form is homogeneous, this means that it is positive definite, and thus Theorem 2 is proved.

The algorithm for covering of the manifold K is appropriately useful in studying the fixed sign property of the forms when their coefficients are specified numerically. In view of the very good analytical properties of forms, other algorithms also suggest themselves for the detection of the positiveness of forms on the manifold K. We choose a finite number U of all the nodes of a grid uniformly marked on the boundaries of the m-dimensional cube $|x| \leqslant a^2$. It is clear that $U \subset K$. Let the length of the side of the smallest cell on the cube boundary be chosen equal to a^2 / ν (ν is an integer). If ν is sufficiently large, then the positiveness of $V(x)$ on U indicates the positiveness of $V(x)$ on K. Consequently, the problem reduces to the calculation of the values of the function $V(x)$ at a finite number of points.

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